ASSIGNMENT

(To be done after studying Blocks 1, 2 and 3)

Course Code: MTE-06
Assignment Code: MTE-06/TMA/2021
Maximum Marks: 100

1) Which of the following statements are true? Justify your answers. (This means that if you think a statement is false, give a short proof or an example that shows it is false. If it is true, give a short proof for saying so.)

   i) If A and B are two sets such that $A \subseteq B$, then $A \times B = B$.

   ii) If S is the set of people on the rolls of IGNOU in 2016 and T is the set of real numbers lying between 2.5 and 2.55, then $S \cup T$ is an infinite set.

   iii) The set $\{x \in \mathbb{Z} \mid x \equiv 1 (\text{mod}\, 30)\}$ is a group with respect to multiplication (mod 30).

   iv) If G is a group with an abelian quotient group G/N, then N is abelian.

   v) There is a group homomorphism f with $\text{Ker } f \simeq \mathbb{R}$ and $\text{Im} f \simeq \{0\}$.

   vi) There is a 1 – 1 correspondence between the odd permutations of S_{35} and the even permutations of S_{35}.

   vii) If R is a ring such that $a = -a \ \forall\, a \in R$, then R is Boolean.

   viii) Given any ring R, there is an ideal I of R such that R/I is commutative.

   ix) If S is an ideal of a ring R and f a ring homomorphism from R to a ring R', then $f^{-1}(f(S)) = S$.

   x) 'ring', as we now define it, was first presented to us by Dedekind. (20)

2) a) Prove that $2^n > 4n$ for $n \geq 5$. (3)

   b) Give an example, with justification, of a function with domain $\mathbb{Z} \setminus \{2,3\}$ and co-domain $\mathbb{N}$. Is this function 1 – 1? Is it onto? Give reasons for your answers. (4)

   c) Give a set of cardinality 5 which is a subset of $\mathbb{Z} \setminus \mathbb{N}$. (1)

   d) Check whether the relation $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid xy \text{ is the square of an integer}\}$ is an equivalence relation on $\mathbb{N}$. (2)

3) a) The table below is a Cayley table for the group $(\{e, a, b, c, d\}, *)$. Fill in the blanks.

*	e	a	b	c	d
e	e	-	-	-	-
a	-	b	-	-	e
b	-	c	d	e	-
c	-	d	-	a	b
d	-	-	-	-	-

(3)

b) Let G be a finite group. Show that the number of elements g of G such that $g^3 = e$ is odd, where e is the identity of G. (3)

c) Check if $\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$ is an abelian group with respect to matrix multiplication. (4)

4) a) Check whether $H = \{x \in \mathbb{R}^* \mid x = 1 \text{ or } x \text{ is irrational}\}$ and $K = \{x \in \mathbb{R}^* \mid x \geq 1\}$ are subgroups of $(\mathbb{R}^*, .)$. (3)

b) Let $U(n) = \{m \in \mathbb{N} \mid (m, n) = 1, m \leq n\}$. Then U(n) is a group with respect to multiplication modulo n. Find the orders of <m> for each $m \in U(10)$. (3)

c) Find $Z(D_{2n})$, where D_{2n} is the dihedral group with 2n elements,

i) when n is an odd integer;
ii) when n is an even integer. (4)

5. a) Obtain the left cosets of $V_4 = \{e, (1\ 2)\ (3\ 4), (1\ 3)\ (2\ 4), (1\ 4)\ (2\ 3)\}$ in A_4. (3)

b) If G is a group with $o(g) < 100$ and G has subgroups of order 10 and 25, what is the order of G? (2)

c) Show that in a group G of odd order, the equation $x^2 = e$ has a unique solution. Further, show that $x^2 = g$ has a unique solution $\forall\, g \in G, g \neq e$. (5)

6. a) Check whether the subgroup of reflections and subgroup of rotations in D_{2n} is normal in D_{2n} or not.
(Note that D_{2n} is the group of symmetries of an n-gon.) (3)

b) Prove, by contradiction, that A_4 has no subgroup of order 6. (3)

c) What is the order of

i) 14 in $\mathbb{Z}_{24} / \langle \overline{8} \rangle$?

ii) $(\mathbb{Z}_{10} \oplus U(10)) / < (2,9) >$? (4)

7. a) Can there be a homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$? Give reasons for your answer. (2)

b) Define $f:\mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n : f(x) = (x \bmod m, x \bmod n), m, n \in \mathbb{N}$.

i) If (m, n) = (3, 4), find Ker f.
ii) If (m, n) = (6, 4), find Ker f.
iii) What can you generalize about Ker f from (i) and (ii)? (3)

c) Use FTH to determine all homomorphic images of D_8, upto isomorphism. (5)

8. a) If U(R) denotes the group of units of a ring R, show that $U(R_1 \times R_2) = U(R_1) \times U(R_2)$ for rings R_1 and R_2. (2)

b) Let R be a ring, I an ideal of R, J an ideal of I. Show that if J has a unity, then J is an ideal of R. Also give an example to show that if J does not satisfy this condition it need not be an ideal of R. (5)

c) Let F be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$ w.r.t. pointwise, addition and multiplication. Let S be the set of all differentiable functions in F. Check whether S is

i) a subring of F,
ii) an ideal of F. (3)

9. a) Prove that every ideal I of a ring R is the kernel of a ring homomorphism of R. (2)

b) Prove that $\mathbb{Z}[\sqrt{2}]$ is isomorphic to $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ as rings. (3)

c) Let R and S be rings and f : R → S be a homomorphism. If x is an idempotent in R, show that f(x) is an idempotent in S. Hence, or otherwise, determine all ring homorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. (5)